Answer Key for Summer Review of Algebra 2 skills

Created to help those entering Precalculus or Honors Precalculus.

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Skill #1: Factoring and Solving Polynomial Equations

- Most common type on Algebra 2 Regents: GROUPING (4 terms)

Factoring Trinomials: You may have to take out a GCF first.

$x^2 - 6x - 16 = (x-8)(x+2)$ add multiply

$2x^3 - 18x^2 + 28x = 2x(x^2 - 9x + 14)$
$= 2x(x-7)(x-2)$

1. $4x^3 - 8x^2 - 32x$
   $4x(x^2 - 2x - 8)$
   $4x(x-4)(x+2)$

2. $3x^3 + 21x^2 + 30x = 0$
   $3x(x^2 + 7x + 10) = 0$
   $3x(x+5)(x+2) = 0$
   $x=0 \quad x=-5 \quad x=-2$

3. $x^2 - 11x = -10$
   $x^2 - 11x + 10 = 0$
   $(x-10)(x-1) = 0$
   $x = 10 \quad x = 1$

Factoring by Grouping: You may need to reorder your expression first or factor out a GCF.

$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1)$ ⇒ Group the 1st two and last two terms & factor the GCF.
$= (2x-1)(x^2+4)$ ⇒ Take out the factor that is the same. The leftovers go in a set of parentheses together. If you can continue to factor, do so. If you can't, stop.

4. $\underbrace{x^3 - 2x^2} + \underbrace{5x - 10}$
   $x^2(x-2) + 5(x-2)$
   $(x-2)(x^2+5)$

5. $3y^4 + 9y^2 - 6y^3 - 18y$ ---- Take out a GCF first.
   $3y(y^3 + 3y - 2y^2 - 6)$
   $3y[y(y^2+3) - 2(y^2+3)]$
   $3y(y^2+3)(y-2)$

6. $4x^4 + 12x^3 + 6x^2 + 18x$
   $2x(\underbrace{2x^3 + 6x^2} + 3x + 9)$
   $2x[2x^2(x+3) + 3(x+3)]$
   $2x(x+3)(2x^2+3)$

7. $\underbrace{3x + 7y} + \underbrace{21 + xy}$
   no GCF! rearrange! $\underbrace{3x + 21} + \underbrace{7y + xy}$
   $3(x+7) + y(7+x)$
   $(x+7)(3+y)$

Factoring Trinomials Special Case: Leading Coefficient Is NOT A GCF

$2x^2 + 9x + 10$ First, multiply the leading coefficient by the last term: $2 \cdot 10 = 20$

- Think of two numbers whose product is 20 and whose sum is 9 (the middle term). The #s are 4 and 5.

$2x^2 + 4x + 5x + 10$ Split the middle term into two terms using those two numbers as coefficients.

$2x(x+2) + 5(x+2)$ Factor by grouping to finish.

$(2x+5)(x+2)$

8. $3x^2 + 10x + 8$ $\quad 3 \cdot 8 = 24$
   $\underbrace{3x^2 + 6x} + \underbrace{4x + 8}$
   $3x(x+2) + 4(x+2)$
   $(x+2)(3x+4)$

9. $8x^2 + 2x - 3$ $\quad 8 \cdot -3 = -24$
   $\underbrace{8x^2 - 6x} + \underbrace{4x - 3}$
   $2x(4x-3) + 1(4x-3)$
   $(4x-3)(2x+1)$

10. $4x^2 - 3x - 10$ $\quad 4 \cdot -10 = -40$
    $\underbrace{4x^2 - 8x} + \underbrace{5x - 10}$
    $4x(x-2) + 5(x-2)$
    $(x-2)(4x+5)$

Skill #2: Polynomial Long Division

- Like regular long division, but with polynomials!

Long Division: Divide, Multiply, Subtract, Bring Down!

Write your answer in the form: $q(x) + \frac{r(x)}{g(x)}$, where $q(x)$ is the quotient, $r(x)$ is the remainder, and $g(x)$ is the divisor.

$$\frac{3x^4-12x+5}{x+1} \quad \Rightarrow \quad \begin{array}{r} 3x^3 - 3x^2 + 3x - 15 \\ x+1 \overline{)\, 3x^4 + 0x^3 + 0x^2 - 12x + 5} \\ \underline{-3x^4 - 3x^3} \\ -3x^3 + 0x^2 \\ \underline{3x^3 + 3x^2} \\ 3x^2 - 12x \\ \underline{-3x^2 - 3x} \\ -15x + 5 \\ \underline{15x + 15} \\ 20 \end{array}$$

This comes from $-(3x^4 + 3x^3)$. I prefer to distribute it out.
Same here. $-(-3x^3 - 3x^2)$

Same here. $-(3x^2 + 3x)$

Same here. $-(-15x - 15)$

So the final answer is: $3x^3 - 3x^2 + 3x - 15 + \frac{20}{x+1}$

1. $\frac{2x^3-5x^2-8x+15}{x-3}$

$$\begin{array}{r} 2x^2 + x - 5 \\ x-3 \overline{)\, 2x^3 - 5x^2 - 8x + 15} \\ \underline{-(2x^3 - 6x^2)} \\ x^2 - 8x \\ \underline{-(x^2 - 3x)} \\ -5x + 15 \\ \underline{-(-5x + 15)} \\ 0 \end{array}$$

$2x^2 + x - 5$

2. $\frac{x^4-8x^3+16x^2-19}{x-5}$

$$\begin{array}{r} x^3 - 3x^2 + x + 5 \\ x-5 \overline{)\, x^4 - 8x^3 + 16x^2 + 0x - 19} \\ \underline{-(x^4 - 5x^3)} \\ -3x^3 + 16x^2 \\ \underline{-(-3x^3 + 15x^2)} \\ x^2 + 0x \\ \underline{-(x^2 - 5x)} \\ 5x - 19 \\ \underline{-(5x - 25)} \\ 6 \end{array}$$

$x^3 - 3x^2 + x + 5 + \frac{6}{x-5}$

Skill #3: Remainder Theorem

- If f(a) = 0, then x-a is a factor. Keep reading to figure out what that means.

If $f(x) = 3x^4 - 12x + 5$ and $g(x) = x + 1$, find the remainder when $\frac{f(x)}{g(x)}$.

Set the denominator = to zero and solve: $x + 1 = 0 \Rightarrow x = -1$

Take your x value and plug it into the original: $f(-1) = 3(-1)^4 - 12(-1) + 5 = 3 + 12 + 5 = 20$

The value you obtain, 20, is the remainder. Since the remainder $\neq 0$, we know $x + 1$ is not a factor.

1. Determine the remainder when $p(x) = x^2 + 7x + 10$ is divided by $d(x) = x + 5$. Then complete the sentence below.

   $p(-5) = (-5)^2 + 7(-5) + 10 = 0$

   $d(x)$ is / is not a factor of $p(x)$, since $p(-5) =$ 0.

   CIRCLE ONE

2. Fill in the blanks: If $g(x)$ is a factor of $f(x)$ and $g(x) = x + 3$, then $f(-3) =$ 0.

3. Fill in the blanks: If $m(x)$ is a factor of $g(x)$ and $m(x) = x - 1$, then $g(1) =$ 0.

4. Determine if $x - 2$ is a factor of $3x^3 - 4x^2 + x - 1$. Explain your answer.

   plug in 2 $3(2)^3 - 4(2)^2 + 2 - 1 = 9$ No - the remainder $\neq 0$

5. Determine if $x + 5$ is a factor of $x^4 - 10x^2 - 375$. Explain your answer.

   plug in -5 $(-5)^4 - 10(-5)^2 - 375 = 0$ Yes - the remainder $= 0$

6. Given $f(x) = 4x^3 + 8x^2 - 3x + 27$, find $f(-3)$. What does your answer tell you?

   $f(-3) = 4(-3)^3 + 8(-3)^2 - 3(-3) + 27$

   $= 0 \rightarrow$ this means $x+3$ is a factor.

7. Given $g(x) = 2x^3 - x + 8$, find $g(2)$. What does your answer tell you?

   $g(2) = 2(2)^3 - 2 + 8$

   $= 22 \rightarrow$ this means $x-2$ is not a factor.

Skill #4: Undefined Fractions

- If the denominator of a fraction equals 0, then the fraction is undefined.

Determine the values that would make the fraction undefined.

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 3x - 10}$$

Set the denominator equal to zero and solve:

$$x^2 - 3x - 10 = 0$$
$$(x - 5)(x + 2) = 0$$
$$x = 5 \quad x = -2$$

So the domain is: $x \neq 5, x \neq -2$ (all real numbers except 5 and -2)

1. Find the values where the fraction is undefined:

$$\frac{x + 5}{x - 2}$$

$x - 2 = 0$

$\boxed{x = 2}$

2. What is the domain of:

$$f(x) = \frac{x^2 + 6x + 8}{x^2 - x - 6}$$

$(x - 3)(x + 2) = 0$

$x = 3 \quad x = -2$

Domain: $x \neq 3, x \neq -2$

3. The fraction $f(x) = \frac{x-3}{x^2-4x-32}$ is undefined when x equals 8, -4.

$(x - 8)(x + 4) = 0$

$x = 8 \quad x = -4$

4. The fraction $f(x) = \frac{x+1}{x^2-4x-5}$ is undefined when x equals 5, -1.

$(x - 5)(x + 1) = 0$

$x = 5 \quad x = -1$

Skill #5: Simplifying Algebraic Fractions

- Apply fraction operations from middle school to algebraic fractions.

Given: $f(x) = \frac{x+2}{x-4}$ and $g(x) = \frac{3}{x-2}$

Multiply $f(x) \cdot g(x) = \frac{x+2}{x-4} \cdot \frac{3}{x-2} = \frac{3x+6}{(x-4)(x-2)}$

Divide $\frac{f(x)}{g(x)} = \frac{x+2}{x-4} \div \frac{3}{x-2} = \frac{x+2}{x-4} \cdot \frac{x-2}{3} = \frac{(x+2)(x-2)}{3(x-4)} = \frac{x^2-4}{3x-12}$

Add $f(x) + g(x) = \frac{x+2}{x-4} + \frac{3}{x-2} = \frac{(x+2)(x-2)}{(x-4)(x-2)} + \frac{3(x-4)}{(x-2)(x-4)} = \frac{x^2-4}{(x-4)(x-2)} + \frac{3x-12}{(x-4)(x-2)} = \frac{x^2+3x-16}{(x-4)(x-2)}$

Subtract $f(x) - g(x) = \frac{x+2}{x-4} - \frac{3}{x-2} = \frac{(x+2)(x-2)}{(x-4)(x-2)} - \frac{3(x-4)}{(x-2)(x-4)} = \frac{x^2-4}{(x-4)(x-2)} - \frac{3x-12}{(x-4)(x-2)} = \frac{x^2-3x+8}{(x-4)(x-2)}$

Simplify:

1. $\frac{x^2-3x-10}{2x^2+8x+8} = \frac{(x-5)\cancel{(x+2)}}{2(x+2)\cancel{(x+2)}} = \boxed{\frac{x-5}{2(x+2)}}$

   $2(x^2+4x+4)$

2. $\frac{4-x^2}{x^2+2x-8} = \frac{\overset{-1}{\cancel{(2-x)}}(2+x)}{(x+4)\cancel{(x-2)}} = \boxed{\frac{-1(2+x)}{x+4}}$

3. $\frac{3x^2+4x-4}{x^2+3x+2} = \frac{(3x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)} = \boxed{\frac{3x-2}{x+1}}$

4. $\frac{ax-a-c+cx}{acx-ac} = \frac{a(x-1)+c(-1+x)}{ac(x-1)}$

   $= \frac{\cancel{(x-1)}(a+c)}{ac\cancel{(x-1)}} = \boxed{\frac{a+c}{ac}}$

Multiply or Divide:

5. $\frac{a+3}{a+2} \cdot \frac{a^2+3a+2}{a^2+4a+3} = \frac{\cancel{a+3}}{\cancel{a+2}} \cdot \frac{\cancel{(a+2)}\cancel{(a+1)}}{\cancel{(a+3)}\cancel{(a+1)}} = \boxed{1}$

6. $\frac{m^2-n^2}{am+an} \div \frac{m-n}{m^2+n^2} = \frac{\cancel{(m+n)}\cancel{(m-n)}}{a\cancel{(m+n)}} \cdot \frac{m^2+n^2}{\cancel{m-n}} = \boxed{\frac{m^2+n^2}{a}}$

Add or Subtract:

7. $\frac{x+3}{x-2} - \frac{x^2}{x^2-4} = \frac{(x+3)(x+2)}{(x+2)(x-2)} - \frac{x^2}{(x+2)(x-2)} = \frac{x^2+5x+6-x^2}{(x+2)(x-2)}$

   $(x+2)(x-2)$

   $= \boxed{\frac{5x+6}{(x+2)(x-2)}}$

8. $2 + \frac{3x}{2-x} = \frac{2(2-x)}{2-x} + \frac{3x}{2-x}$

   $= \frac{4-2x+3x}{2-x} = \boxed{\frac{4-x}{2-x}}$

Skill #6: Solving Rational Equations

- Get a common denominator then set the numerators equal. Check for values that make the original fractions undefined!

Undefined at $x = -2$ and $x = 3$

$$\frac{2}{x+2} - \frac{x+3}{x-3} = \frac{4x-1}{x^2-x-6}$$

$$\frac{2(x-3)}{(x+2)(x-3)} - \frac{(x+3)(x+2)}{(x+2)(x-3)} = \frac{4x-2}{(x+2)(x-3)}$$

$$\frac{2x-6}{(x+2)(x-3)} - \frac{x^2+5x+6}{(x+2)(x-3)} = \frac{4x-2}{(x+2)(x-3)}$$

$$2x - 6 - x^2 - 5x - 6 = 4x - 2$$

$$-x^2 - 3x - 12 = 4x - 2$$

$$-x^2 - 7x - 10 = 0$$

$$x^2 + 7x + 10 = 0$$

$$(x+5)(x+2) = 0$$

$$\boxed{x = -5} \quad \cancel{x = -2}$$

Reject since $x \neq -2$ since it would make the original fraction undefined

1. $\frac{x}{x^2-1} + \frac{2}{x+1} = \frac{1}{2x-2}$

$(x+1)(x-1)$ $\quad 2(x-1)$

$$\frac{2 \cdot x}{2(x+1)(x-1)} + \frac{2 \cdot 2(x-1)}{2(x+1)(x-1)} = \frac{1 \cdot (x+1)}{2(x-1)(x+1)}$$

$$2x + 4x - 4 = x + 1$$

$$6x - 4 = x + 1$$

$$5x = 5$$

$$\cancel{x = 1}$$

$\boxed{\text{reject} \rightarrow \text{no solution}}$

2. $\frac{x-3}{x-7} - \frac{1}{x} = \frac{28}{x^2-7x}$

$x(x-7)$

$$\frac{x(x-3)}{x(x-7)} - \frac{1(x-7)}{x(x-7)} = \frac{28}{x(x-7)}$$

$$x^2 - 3x - x + 7 = 28$$

$$x^2 - 4x + 21 = 0$$

$$(x-7)(x+3) = 0$$

$$\cancel{x = 7} \quad \boxed{x = -3}$$

3. $\frac{1}{2x} - \frac{9}{x^2+6x} = \frac{2-x}{2x+12}$

$x(x+6)$ $\quad 2(x+6)$

$$\frac{1(x+6)}{2x(x+6)} - \frac{9 \cdot 2}{2x(x+6)} = \frac{x(2-x)}{2x(x+6)}$$

$$x + 6 - 18 = 2x - x^2$$

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

$$\boxed{x = 4 \quad x = -3}$$

Skill #7: Exponent Rules

- You know… all the rules from Algebra 1.

Multiplying: *Add Exponents* $\Rightarrow x^2 \cdot x^3 = x^5$ Dividing: *Subtract Exponents* $\Rightarrow \frac{x^6}{x^2} = x^4$

Zero Exponent: *Equals 1* $\Rightarrow x^0 = 1 \quad (3x)^0 = 1$ Power to A Power: *Multiply Exponents* $\Rightarrow (x^2)^3 = x^6$

Negative Exponents: *Become Fractional* $\Rightarrow x^{-2} = \frac{1}{x^2}$

Simplify:

1. $3x^3 \cdot x^9 = 3x^{12}$
2. $(x^2y^3)(x^5y)$

   x^7y^4
3. $\frac{12x^6}{3x^3}$

   $4x^3$
4. $-\frac{10a^6b^2c}{5a^2bc}$

   $-2a^4b$
5. $(y^{-5})^{-3}$

   y^{15}
6. $(3^a)^b$

   3^{ab}
7. $(2x^3)^4$

   $16x^{12}$
8. $(-m^2)^5$

   $-m^{10}$
9. $\left(\frac{3}{a^2}\right)^3$

   $\frac{27}{a^6}$
10. $3x^0$

    3
11. $(3x)^0$

    1
12. x^{-3}

    $\frac{1}{x^3}$
13. $6x^{-3}$

    $\frac{6}{x^3}$
14. a^2b^{-3}

    $\frac{a^2}{b^3}$
15. $\frac{x^{-2}}{x}$

    $\frac{1}{x^3}$
16. $\frac{x^2y^{-3}}{x^{-3}y^{-2}}$

    $x^5y^{-1} = \frac{x^5}{y}$
17. $\frac{a^4b^{-3}}{ab^{-2}}$

    $a^3b^{-1} = \frac{a^3}{b}$
18. $\frac{3a^{-3}}{6b^{-2}}$

    $\frac{b^2}{2a^3}$
19. $\left(\frac{x^4y^{-2}}{x^5y^{-3}}\right)^{-2} = \frac{x^{-8}y^4}{x^{-10}y^6} = \frac{x^2}{y^2}$
20. $\frac{(2a^2b^4)^2}{2a^3b^{-5}} = \frac{4a^4b^8}{2a^3b^{-5}} = 2ab^{13}$

Skill #8: Rational Exponents ↔ Radical Form

- A rational (fractional) exponent can be converted into radical form:

$$x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

The ROOT is in the DENOMINATOR (bottom)! Just like the root of a tree is in the ground (bottom)!

Rewrite $x^{\frac{2}{3}}$ as a radical: $\sqrt[3]{x^2}$ OR $\left(\sqrt[3]{x}\right)^2$ Rewrite $\sqrt[4]{a^3}$ as a rational exponent: $a^{\frac{3}{4}}$

Complete the Chart: The first one has been completed for you.

Radical	Fractional (Rational) Exponent	Simplified Form
$\sqrt{4}$	$4^{\frac{1}{2}}$	2
$\sqrt[3]{8}$	$8^{\frac{1}{3}}$	2
$\sqrt{16}$	$16^{\frac{1}{2}}$	4
$\sqrt[4]{16}$	$16^{\frac{1}{4}}$	2
$\left(\sqrt[3]{64}\right)^2$	$64^{2/3}$	16
$\sqrt[4]{81^3}$ OR $\left(\sqrt[4]{81}\right)^3$	$81^{\frac{3}{4}}$	27
$\frac{1}{\sqrt[4]{81^3}}$ OR $\frac{1}{\left(\sqrt[4]{81}\right)^3}$	$81^{-\frac{3}{4}}$	$1/27$
$\frac{1}{\sqrt{25}}$	$25^{-\frac{1}{2}}$	$1/5$

Rewrite each radical expression as a power with a fractional exponent:

1. $\sqrt[3]{12}$

   $12^{\frac{1}{3}}$

2. $\sqrt[5]{x^4}$

   $x^{4/5}$

3. $\left(\sqrt[5]{2}\right)^3$

   $2^{3/5}$

4. $x\sqrt{y^3}$

   $x \cdot y^{3/2}$

Rewrite with a radical sign instead of a fractional exponent:

5. $x^{\frac{1}{4}}$

   $\sqrt[4]{x}$

6. $xy^{\frac{1}{4}}z^{\frac{3}{4}}$

   $x\sqrt[4]{yz^3}$

7. $n^{\frac{2}{3}}$

   $\sqrt[3]{n^2}$

8. $ab^{\frac{1}{2}}$

   $a\sqrt{b}$

Rewrite with fractional exponents (if necessary) and simplify.

9. $x^{\frac{1}{5}} \cdot x^{\frac{2}{5}}$

   $x^{3/5}$

   $\sqrt[5]{x^3}$

10. $\frac{x^{\frac{7}{5}}}{x^{\frac{3}{4}}}$

    $x^{13/20}$

    $\sqrt[20]{x^{13}}$

11. $\sqrt{x} \cdot \sqrt[3]{x}$

    $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$

    $x^{5/6}$

    $\sqrt[6]{x^5}$

12. $\frac{\sqrt[3]{x^2}}{\sqrt[6]{x}} = \frac{x^{\frac{2}{3}}}{x^{\frac{1}{6}}}$

    $= x^{\frac{1}{2}}$

    $\sqrt{x}$

Skill #9: Radical Equations

- Isolate the radical, then square both sides and solve. Check your answer! (Usually one answer gets rejected.)

$\sqrt{-x-1}+x=4x+5$

$\sqrt{-x-1}=3x+5$ Get the square root by itself.

$\left(\sqrt{-x-1}\right)^2=(3x+5)^2$ Square both sides.

$-x-1=(3x+5)(3x+5)$ Simplify.

$-x-1=9x^2+30x+25$ Simplify.

$9x^2+31x+26=0$ Set equal to zero.

$9x^2+13x+18x+26=0$ Factor (using grouping on this one).

$x(9x+13)+2(9x+13)=0$

$(9x+13)(x+2)=0$

$x=-\frac{13}{9}$ $\cancel{x=-2}$ Solve and check.

REJECT

Check:

$x=-\frac{13}{9}\Rightarrow$

$\sqrt{-\left(-\frac{13}{9}\right)-1}+\left(-\frac{13}{9}\right)=4\left(-\frac{13}{9}\right)+5$

$-0.\overline{7}=-0.\overline{7}$

So $x=-\frac{13}{9}$ is a solution.

$x=-2\Rightarrow$

$\sqrt{-(-2)-1}+(-2)=4(-2)+5$

$-2\neq-3$

So $x=-2$ is NOT a solution.

REJECT $x=-2$.

Solve each equation for x. Make sure you verify your solution by checking!

1. $x=1+\sqrt{x+5}$

$(x-1)^2=(\sqrt{x+5})^2$

$x^2-2x+1=x+5$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$\boxed{x=4}$ $\cancel{x=-1}$

2. $(\sqrt{x^2-6x})^2=(4)^2$

$x^2-6x=16$

$x^2-6x-16=0$

$(x-8)(x+2)=0$

$\boxed{x=8 \quad x=-2}$

3. $\sqrt{2x-7}-5=-x$

$(\sqrt{2x-7})^2=(-x+5)^2$

$2x-7=(-x+5)(-x+5)$

$2x-7=x^2-10x+25$

$x^2-12x+32=0$

$(x-8)(x-4)=0$

$\cancel{x=8}$ $\boxed{x=4}$

4. $3\sqrt{x-2}-2\sqrt{x+8}=0$

Hint: Move one square root over to the other side first (so that you have one square root on each side. Then square both sides.)

$(3\sqrt{x-2})^2=(2\sqrt{x+8})^2$

$9(x-2)=4(x+8)$

$9x-18=4x+32$

$5x=50$

$\boxed{x=10}$

Skill #10: Complex Number Operations

- You don't need to know this section. It's not real...... jk → $i^2 = -1$

Simplify: $4xi^2(-8xi - 2) = -16x^2i^3 - 8xi^2$ Distribute.

$= -16x^2(-i) - 8x(-1)$ Simplify: $i^3 = -i$ and $i^2 = -1$

$= 16x^2i + 8x$ Multiply.

<u>Note:</u> $i^0 = 1$ $i^1 = i$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$

CALCULATOR should be in $a + bi$ MODE!!

Simplify completely.

1. $ki(-k^2 + 3i)$

$-k^3i + 3ki^2$

$-k^3i + 3k(-1)$

$\boxed{-k^3i - 3k}$

2. $2mi(i^2 + m)$

$2mi^3 + 2m^2i$

$2m(-i) + 2m^2i$

$\boxed{-2mi + 2m^2i}$

3. $(x + 3i)^2$

$(x+3i)(x+3i)$

$x^2 + 3xi + 3xi + 9\overset{-1}{i^2}$

$\boxed{x^2 + 6xi - 9}$

4. $(i - 5i)^2$

$(-4i)^2 = 16i^2$

$= \boxed{-16}$

4. $2xi^3(5 + 2xi)$

$10xi^3 + 4x^2i^4$

$10x(-i) + 4x^2$

$\boxed{-10xi + 4x^2}$

6. $-7i(i^2 - 7i^2)^2$

$-7i(-6i^2)^2$

$-7i(36i^4)$

$-7i(36)$

$\boxed{-252i}$

7. $(1 - i)^3$

$(1-i)(1-i)(1-i)$

$(1-2i+i^2)(1-i)$

$(1-2i-1)(1-i)$

$-2i(1-i)$

$-2i + 2i^2$

$\boxed{-2i - 2}$

8. $(1 - xi)^3$

$(1-xi)(1-xi)(1-xi)$

$(1 - xi - xi + x^2i^2)(1-xi)$

$(1-2xi-x^2)(1-xi)$

$1 - \underline{\underline{2xi}} - x^2 - \underline{\underline{xi}} + 2x^2i^2 + x^3i$

$1 - 3xi - x^2 - 2x^2 + x^3i$

$\boxed{1 - 3xi - 3x^2 + x^3i}$

Skill #11: Quadratics with Complex Solutions

- When you use the quadratic formula, you might get a negative under the radical. This section shows you how to deal with that.

Solve for x: $2x^2 + 4x + 7 = 0$ Can't be factored, so use the quadratic formula.

$a = 2, b = 4, c = 7$

$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(7)}}{2(2)}$ Plug the values into the formula.

$x = \frac{-4 \pm \sqrt{-40}}{4}$ Simplify.

$x = \frac{-4 \pm \sqrt{-4}\sqrt{10}}{4}$ Break down the radical.

$x = \frac{-4 \pm 2i\sqrt{10}}{4}$ Simplify the radical.

$x = -\frac{4}{4} \pm \frac{2i\sqrt{10}}{4}$ Divide each term by the denominator.

$x = -1 \pm \frac{1}{2} i\sqrt{10}$ Simplify. This way of writing your answer is called $a + bi$ form.

Solve for x. Leave your answer in $a + bi$ form. (Note: This does <u>not</u> mean to only indicate the answer with the "+" sign. Technically, it means to leave your answer in $a \pm bi$ form.)

1. $8x^2 - 4x + 5 = 0$

$x = \frac{4 \pm \sqrt{(-4)^2 - 4(8)(5)}}{2(8)}$

$= \frac{4 \pm \sqrt{-144}}{16} = \frac{4 \pm 12i}{16}$

$= \boxed{\frac{1}{4} \pm \frac{3}{4}i}$

2. $6x^2 + 2x = -5$

$6x^2 + 2x + 5 = 0$

$x = \frac{-2 \pm \sqrt{2^2 - 4(6)(5)}}{2(6)}$

$= \frac{-2 \pm \sqrt{-116}}{12} = \frac{-2 \pm \sqrt{-4}\sqrt{29}}{12}$

$= \frac{-2 \pm 2i\sqrt{29}}{12} = \boxed{-\frac{1}{6} \pm \frac{1}{6} i\sqrt{29}}$

3. $2x^2 = -6x - 9$

$2x^2 + 6x + 9 = 0$

$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(9)}}{2(2)}$

$= \frac{-6 \pm \sqrt{-36}}{4} = \frac{-6 \pm 6i}{4}$

$= \boxed{-\frac{3}{2} \pm \frac{3}{2}i}$

4. $x^2 + 6x + 12 = 0$

$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)}$

$= \frac{-6 \pm \sqrt{-12}}{2} = \frac{-6 \pm \sqrt{-4}\sqrt{3}}{2}$

$= \frac{-6 \pm 2i\sqrt{3}}{2} = \boxed{-3 \pm i\sqrt{3}}$

Skill #12: Systems of Equations – 2x2

- Basic. These ones are not usually explicitly questioned on the Algebra 2 Regents, but they're necessary to understand how to do 3x3.

Solving by elimination:

$-x + 5y = 8 \Rightarrow$ Multiply by 3 $\Rightarrow 3(-x + 5y = 8) \Rightarrow -\cancel{3x} + 15y = 24$

$3x + 7y = -2 \Rightarrow$ Leave the same $\Rightarrow 3x + 7y = -2 \Rightarrow \cancel{3x} + 7y = -2$

Add them together: $22y = 22$

Solve: $y = 1$

Using $y = 1$, solve for x.

You can use either ORIGINAL equation! $-x + 5(1) = 8 \Rightarrow -x + 5 = 8 \Rightarrow -x = 3 \Rightarrow x = -3$

Final solution: $x = -3, y = 1$ so $(-3, 1)$ is the final solution.

Solve each system for x and y.

1. $3(5x - 2y = -19)$
   $2(2x + 3y = 0)$

$15x - \cancel{6y} = -57$
$4x + \cancel{6y} = 0$

$19x = -57$
$x = -3$

$2(-3) + 3y = 0$
$-6 + 3y = 0$
$3y = 6$
$y = 2$

$(-3, 2)$

2. $\frac{1}{2}x + \frac{2}{3}y = 1$
   $2\left(\frac{3}{4}x - \frac{1}{3}y = 2\right)$

$\frac{1}{2}x + \cancel{\frac{2}{3}y} = 1$
$\frac{3}{2}x - \cancel{\frac{2}{3}y} = 4$

$2x = 5$
$x = \frac{5}{2}$

$\frac{1}{2}\left(\frac{5}{2}\right) + \frac{2}{3}y = 1$
$\frac{5}{4} + \frac{2}{3}y = 1$
$\frac{2}{3}y = -\frac{1}{4}$
$y = -\frac{3}{8}$

$\left(\frac{5}{2}, -\frac{3}{8}\right)$

3. $2(5x - 3y = -1)$
   $3(3x + 2y = 7)$

$10x - \cancel{6y} = -2$
$9x + \cancel{6y} = 21$

$19x = 19$
$x = 1$

$5(1) - 3y = -1$
$5 - 3y = -1$
$-3y = -6$
$y = 2$

$(1, 2)$

4. $3(4x - 7y = 2)$
   $-4(3x - 3y = 6)$

$\cancel{12x} - 21y = 6$
$-\cancel{12x} + 12y = -24$

$-9y = -18$
$y = 2$

$4x - 7(2) = 2$
$4x - 14 = 2$
$4x = 16$
$x = 4$

$(4, 2)$

Skill #13: Systems of Equations – 3x3

- 3 equations, 3 unknowns – take it slow to avoid mistakes.

Solve by elimination:

$x + 2y + z = 10$ Equation 1
$2x - y + 3z = -5$ Equation 2
$2x - 3y - 5z = 27$ Equation 3

Step 1: Choose two equations and eliminate one variable. I choose equations 2 & 3. I will multiply equation 2 by -1 and eliminate the x's.

$$\begin{aligned} -\cancel{2x} + y - 3z &= 5 \\ \cancel{2x} - 3y - 5z &= 27 \\ \hline -2y - 8z &= 32 \end{aligned}$$

Step 2: Choose two more equations and eliminate the SAME variable as in Step 1. I choose equations 1 & 2. I will multiply equation 1 by -2.

$$\begin{aligned} -\cancel{2x} - 4y - 2z &= -20 \\ \cancel{2x} - y + 3z &= -5 \\ \hline -5y + z &= -25 \end{aligned}$$

$$\begin{aligned} -2y - \cancel{8z} &= 32 \\ -40y + \cancel{8z} &= -200 \\ \hline -42y &= -168 \end{aligned}$$

$\boxed{y = 4}$

Step 3: Take the two new equations and eliminate a variable. I will eliminate z's by multiplying $-5y + z = -25$ by 8. Once I get my solution for y, I will use one of these two equations again to find z.

$-2y - 8z = 32$
$-2(4) - 8z = 32$
$-8 - 8z = 32$
$-8z = 40$
$\boxed{z = -5}$

Step 4: Now that you have solutions for y and z, use one of the original equations to solve for x. I will use equation 1.

$x + 2y + z = 10$
$x + 2(4) - 5 = 10$
$x + 3 = 10$
$\boxed{x = 7}$

Final Solution: $(7, 4, -5)$

Solve each system for the three given variables.

1. $3x - 2y + 4z = 20$
   $-x + 5y + 12z = 73$
   $x + 3y - 2z = 1$

$$\begin{aligned} -\cancel{x} + 5y + 12z &= 73 \\ \cancel{x} + 3y - 2z &= 1 \\ \hline 8y + 10z &= 74 \end{aligned}$$

$$\begin{aligned} \cancel{3x} - 2y + 4z &= 20 \\ -\cancel{3x} + 15y + 36z &= 219 \\ \hline 13y + 40z &= 239 \end{aligned}$$

$\times -4$

$$\begin{aligned} -32y - \cancel{40z} &= -296 \\ 13y + \cancel{40}z &= 239 \\ \hline -19y &= -57 \end{aligned}$$

$\boxed{y = 3}$

$8(3) + 10z = 74$
$24 + 10z = 74$
$10z = 50$
$\boxed{z = 5}$

$x + 3(3) - 2(5) = 1$
$x + 9 - 10 = 1$
$x - 1 = 1$
$\boxed{x = 2}$

$(2, 3, 5)$

2. $x + 2y + 3x = 17$
$-4x + 2y - z = 24$
$3x - 6y - 8x = -67$

$$\begin{aligned} -x - \cancel{2y} - 3z &= -17 \\ -4x + \cancel{2y} - z &= 24 \\ \hline -5x - 4z &= 7 \end{aligned}$$

$$\begin{aligned} 3x + \cancel{6y} + 9z &= 51 \\ 3x - \cancel{6y} - 8z &= -67 \\ \hline 6x + z &= -16 \end{aligned}$$

$\times 4$

$$\begin{aligned} -5x - \cancel{4z} &= 7 \\ 24x + \cancel{4z} &= -64 \\ \hline 19x &= -57 \end{aligned}$$

$\boxed{x = -3}$

$6(-3) + z = -16$
$-18 + z = -16$
$\boxed{z = 2}$

$-3 + 2y + 3(2) = 17$
$-3 + 2y + 6 = 17$
$2y + 3 = 17$
$2y = 14$
$\boxed{y = 7}$

$(-3, 7, 2)$

3. $4x + 2y - 2z = 10$
$2x + 8y + 4z = 32$
$30x + 12y - 4z = 24$

$$\begin{aligned} 2x + 8y + \cancel{4z} &= 32 \\ 30x + 12y - \cancel{4z} &= 24 \\ \hline 32x + 20y &= 56 \end{aligned}$$

$$\begin{aligned} 8x + 4y - \cancel{4z} &= 20 \\ 2x + 8y + \cancel{4z} &= 32 \\ \hline 10x + 12y &= 52 \end{aligned}$$

$\downarrow \times 3$

$\times -5$

$$\begin{aligned} 96x + \cancel{60y} &= 168 \\ -50x - \cancel{60y} &= -260 \\ \hline 46x &= -92 \end{aligned}$$

$\boxed{x = -2}$

$10(-2) + 12y = 52$
$-20 + 12y = 52$
$12y = 72$
$\boxed{y = 6}$

$4(-2) + 2(6) - 2z = 10$
$-8 + 12 - 2z = 10$
$4 - 2z = 10$
$-2z = 6$
$\boxed{z = -3}$

$(-2, 6, -3)$

Skill #14: Systems of Equations – Circle/Quadratic and Line

- Where do a circle and a line intersect? Where do a parabola and a line intersect? Burning questions, I know.

Solve the following system of equations algebraically:

$(x-4)^2 + (y-3)^2 = 25$ (Circle)

$3x + 4y = 24$ (Line)

Step 1: Solve the line for x or y, whichever is easier to you. I'm going to solve for y.

$$4y = -3x + 24$$
$$y = -\frac{3}{4}x + 6$$

Step 2: Substitute your equation from Step 1 into the appropriate location in the circle equation.

$$(x-4)^2 + \left(-\frac{3}{4}x + 6 - 3\right)^2 = 25$$

Step 3: Simplify and solve.

$$(x-4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 25$$
$$(x-4)(x-4) + \left(-\frac{3}{4}x + 3\right)\left(-\frac{3}{4}x + 3\right) = 25$$
$$x^2 - 8x + 16 + \frac{9}{16}x^2 - 4.5x + 9 = 25$$
$$1.5625x^2 - 12.5x + 25 = 25$$
$$1.5625x^2 - 12.5x = 0$$
$$x^2 - 8x = 0 \quad \text{Divide by 1.5625.}$$
$$x(x-8) = 0$$
$$x = 0,\ x = 8$$

Solve for y: Substitute your x values into either of your original equations.

$x = 0$: $3(0) + 4y = 24$ | $x = 8$: $3(8) + 4y = 24$

$4y = 24$ | $24 + 4y = 24$

$y = 6$ | $y = 0$

Final Solutions: $(0, 6)$ and $(8, 0)$ <-- Note: Your final answers will NOT always have zero in them.

Solve each system algebraically.

1. $x^2 + y^2 = 100$
   $y - x = 2$

$y = x + 2$

$x^2 + (x+2)^2 = 100$

$x^2 + x^2 + 4x + 4 = 100$

$2x^2 + 4x - 96 = 0$

$x^2 + 2x - 48 = 0$

$(x+8)(x-6) = 0$

$x = -8$ $\quad$ $x = 6$

$y = -8 + 2$ $\quad$ $y = 6 + 2$

$y = -6$ $\quad$ $y = 8$

$\boxed{(-8, -6) \text{ and } (6, 8)}$

2. $2x + y = 15 \longrightarrow y = -2x + 15$
$(x-2)^2 + (y-1)^2 = 25$

$$(x-2)^2 + (-2x+15-1)^2 = 25$$
$$x^2 - 4x + 4 + (-2x+14)^2 = 25$$
$$x^2 - 4x + 4 + 4x^2 - 56x + 196 = 25$$
$$5x^2 - 60x + 175 = 0$$
$$x^2 - 12x + 35 = 0$$
$$(x-7)(x-5) = 0$$
$$x = 7 \qquad x = 5$$
$$y = -2(7) + 15 \qquad y = -2(5) + 15$$
$$y = 1 \qquad y = 5$$
$$\boxed{(7,1) \text{ and } (5,5)}$$

3. $6x - 3 = y + x^2$
$7 = x + y \rightarrow y = -x + 7$

$$6x - 3 = -x + 7 + x^2$$
$$x^2 - 7x + 10 = 0$$
$$(x-5)(x-2) = 0$$
$$x = 5 \qquad x = 2$$
$$y = -5 + 7 \qquad y = -2 + 7$$
$$y = 2 \qquad y = 5$$
$$\boxed{(5,2) \text{ and } (2,5)}$$

4. $y + 4x = \cancel{y} - x^2 + 12$
$4 = 2x + y \longrightarrow y = -2x + 4$

$$4x = -x^2 + 12$$
$$x^2 + 4x - 12 = 0$$
$$(x+6)(x-2) = 0$$
$$x = -6 \qquad x = 2$$
$$y = -2(-6) + 4 \qquad y = -2(2) + 4$$
$$y = 16 \qquad y = 0$$
$$\boxed{(-6,16) \text{ and } (2,0)}$$

Skill #15: Systems of Equations – Random (Calculator Skill)

- Combo move: 2^{nd} – Trace – Intersect

Tips and Tricks:

- How can you tell you might need to use your calculator?
  (1) They ask for where $f(x) = g(x)$ in a multiple choice question.
  (2) It's not an equation you know how to solve (like a log on one side and an absolute value on the other side).
  (3) It asks you to *round.* This means the intersection point isn't a whole number, which means it may not easily be solvable. (Make sure you round correctly!!)
- The solution is always the x-value of the intersection **unless they specifically ask for the point of intersection.** Sometimes the intersection points might be out of the standard view!
- You cannot use your calculator on a question in which you are told to solve algebraically. If told that you must solve algebraically, you could use this method to check. It would not be enough for full credit.

If $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = |x+5| - 2$, determine where $f(x) = g(x)$ to the *nearest tenth.*

Solution: Using your calculator: $y_1 = \left(\frac{1}{3}\right)^x$ and $y_2 = |x+5| - 2$.
Graph and press 2nd – Trace – 5:Intersect
"First curve?" Arrow to the intersection point and hit enter.
"Second curve?" Arrow to the intersection pint and hit enter.
"Guess?" Hit enter.
Intersection: $x = -0.741552$ $y = 02.2584482$
Rounded to the *nearest tenth:* $x = -0.7$ -- Notice the y-value is not part of the solution.

Using your calculator, answer the following.

1. Determine the solution(s) to the system $f(x) = x^3 - 2x + 7$ and $g(x) = 2x + 5$.

   $x = -2.21432$ $x = .5391889$ $x = 1.6751309$

2. Find the solutions of $g(x) = h(x)$ if $g(x) = |2x + 3|$ and $h(x) = x^3$.

   $x = 1.8932892$

3. Find where $2^{x+3} = x + 5$. $x = -4.690093$

   $x = -1$

4. Determine the points where $f(x) = k(x)$ if $f(x) = 3^x - 7$ and $k(x) = 3x^4 - 8$.

   $(-.826977, -6.596882)$
   $(1.0964058, -3.664826)$

Skill #16: Average Rate of Change

- Just another phrase that means slope!

Slope Formula: Not on the reference sheet!!

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given the table that shows inches of snowfall each day of a particular week, find the average rate of change between day 1 and day 4 and explain what it means in context.

Days	1	2	3	4	5	6	7
Snowfall	4.2	2	6.1	0.5	3	7.2	1

$$m = \frac{4.2 - 0.5}{1 - 4} = \frac{3.7}{-3} = -1.2\overline{3}$$

It means that the amount of snow that fell between day 1 and day 4 is decreasing on average by $1.2\overline{3}$ inches per day.

Using the slope formula, answer the following questions.

1. Which of the following functions has the largest average rate of change on the interval $[-3, 0]$?

$f(x) = |2x + 1| - 3$

x	$g(x)$
-3	-4
-2	3
-1	-6
0	2
1	-8

$(-3, 2)$ $(0, -2)$

$m = \frac{-2-2}{0--3} = \boxed{\frac{-4}{3}}$

$m = \frac{-4-2}{-3-0} = \frac{-6}{-3} = \boxed{2}$

g(x) has the larger avg. rate of change.
$2 > -4/3$

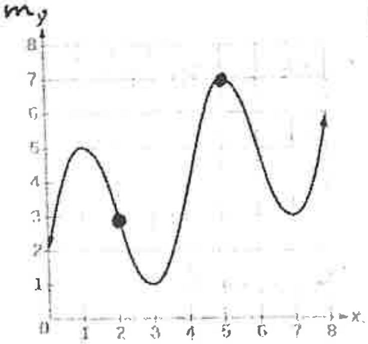
2. The table shows the average diameter of a pupil in a person's eye as he or she grows older. Find the average rate of change from age 20 to age 80. Explain what this average rate of change means in context.

Age (years)	Average Pupil Diameter (mm)
20	4.7
30	4.3
40	3.9
50	3.5
60	3.1
70	2.7
80	2.3

$m = \frac{4.7-2.3}{20-80} = \frac{2.4}{-60} = -.04$

Each yr, the avg pupil diameter decreases by .04 mm.

3. Determine the average rate of change from $x = 2$ to $x = 5$.



$(2, 3)$ $(5, 7)$

$m = \frac{7-3}{5-2} = \boxed{\frac{4}{3}}$

Skill #17: Algebra of Functions

- Basic operations with polynomials as functions.

If $f(x) = 4x^2 + 2x - 1$ and $g(x) = 2x$, find:

- $f(x) + g(x) = 4x^2 + 2x - 1 + 2x = 4x^2 + 4x - 1$
- $g(x) - f(x) = 2x - (4x^2 + 2x - 1) = 2x - 4x^2 - 2x + 1 = -4x^2 + 1$
- $g(x) \cdot f(x) = 2x(4x^2 + 2x - 1) = 8x^3 + 4x^2 - 2x$
- $\frac{f(x)}{g(x)} = \frac{4x^2+2x-1}{2x} = \frac{4x^2}{2x} + \frac{2x}{2x} - \frac{1}{2x} = 2x + 1 - 0.5x^{-1}$

Simplify each of the following.

1. If $f(x) = 3x - 2$ and $g(x) = 5x + 7$, find $2f(x) + g(x)$.

$2(3x-2) + 5x + 7$

$6x - 4 + 5x + 7$

$\boxed{11x + 3}$

2. If $f(x) = x^2 + 7x + 10$ and $g(x) = x^3$, find $\frac{f(x)}{g(x)}$.

$\frac{x^2 + 7x + 10}{x^3} = \frac{x^2}{x^3} + \frac{7x}{x^3} + \frac{10}{x^3} = \boxed{x^{-1} + 7x^{-2} + 10x^{-3}}$

3. If $f(x) = x + 1$ and $g(x) = 2x - 3$, find:

a. $f(x) \cdot g(x)$

$(x+1)(2x-3)$

$2x^2 - 3x + 2x - 3$

$\boxed{2x^2 - x - 3}$

b. $2[f(x) + 1]^2 - 3$

$2[x+1+1]^2 - 3$

$2(x+2)^2 - 3$

$2(x^2 + 4x + 4) - 3$

$2x^2 + 8x + 8 - 3$

$\boxed{2x^2 + 8x + 5}$

c. $[g(x)]^2 + 5$

$(2x-3)^2 + 5$

$(2x-3)(2x-3) + 5$

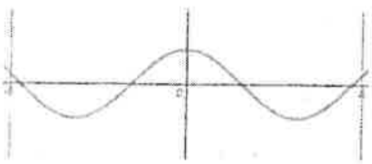
$4x^2 - 12x + 9 + 5$

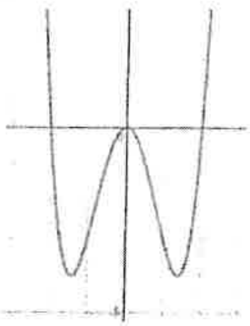
$\boxed{4x^2 - 12x + 14}$

Skill #18: Special Functions – Even/Odd, Inverses

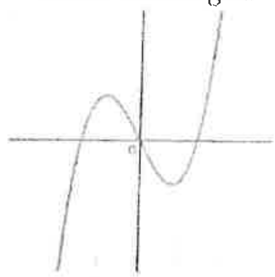
- Even Functions: Symmetric about the y-axis.
- Odd Functions: Symmetric about the origin.
- Inverse Functions ($f^{-1}(x)$): Switch x and y, then solve for y.

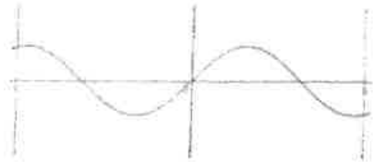
<u>Even Functions</u>: These functions have a line of symmetry on the y-axis.





<u>Odd Functions:</u> These functions are symmetric about the origin.





Algebraically: Classify each function as even, odd, or neither.

If even: $f(-x) = f(x)$
$f(-x)$ is the SAME as the original.

If odd: $f(-x) = -f(x)$
$f(-x)$ is the OPPOSITE SIGNS of the original.

If neither: Neither is true.

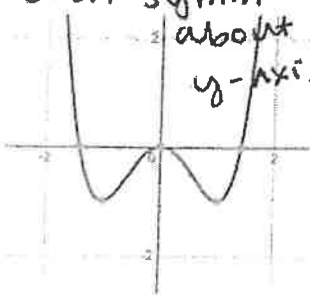
(1) $f(x) = 2x^3 + x$
$f(-x) = 2(-x)^3 + (-x)$
$f(-x) = -2x^3 - x$
$f(-x) = -f(x)$
So $f(x)$ is ODD.

(2) $f(x) = -3x^2 - x$
$f(-x) = -3(-x)^2 - (-x)$
$f(-x) = -3x^2 + x$
$f(-x)$ is not $f(x)$ nor $-f(x)$.
So $f(x)$ is NEITHER.

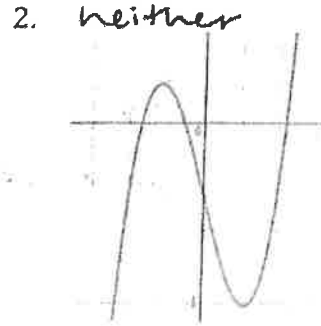
(3) $f(x) = \frac{1}{2}x^4 - 2x^2 + 4$
$f(-x) = \frac{1}{2}(-x)^4 - 2(-x)^2 + 4$
$f(-x) = \frac{1}{2}x^4 - 2x^2 + 4$
$f(-x) = f(x)$
So $f(x)$ is EVEN.

Classify each of the following as even, odd, or neither. Explain/justify your answer.

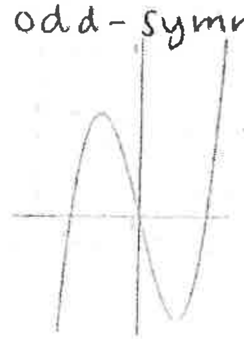
1. even symm. about y-axis.



2. neither



3. odd - symm. about the origin.



4. $f(x) = 3x^3 - 7x$

$f(-x) = 3(-x)^3 - 7(-x)$
$= -3x^3 + 7x$
$= -f(x)$
$\therefore$ odd

5. $g(x) = 3x^8 + 4x^2 - x$

$g(-x) = 3(-x)^8 + 4(-x)^2 - (-x)$
$= 3x^8 + 4x^2 + x$

$\therefore$ neither

Find the inverse of:

a) $f(x) = 2x + 5$

$x = 2y + 5$

$x - 5 = 2y$

$y = \frac{x-5}{2}$

$y = \frac{1}{2}x - \frac{5}{2}$

$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

b) $g(x) = \sqrt[3]{x-3}$

$x = \sqrt[3]{y-3}$ Switch x and y.

$x^3 = \left(\sqrt[3]{y-3}\right)^3$ Solve for y.

$x^2 = y - 3$

$y = x^2 + 3$

$g^{-1}(x) = x^2 + 3$ Replace y using inverse notation.

Find the inverse of each of the following.

1. $f(x) = 2x + 12$

$x = 2y + 12$

$x - 12 = 2y$

$y = \frac{x-12}{2}$

$\boxed{f^{-1}(x) = \frac{x-12}{2}}$

OR $f^{-1}(x) = \frac{1}{2}x - 6$

2. $g(x) = \sqrt{x+4}$

$(x)^2 = (\sqrt{y+4})^2$

$x^2 = y + 4$

$y = x^2 - 4$

$\boxed{f^{-1}(x) = x^2 - 4}$

3. $k(x) = 3(x-2) + 4$

$x = 3(y-2) + 4$

$x = 3y - 6 + 4$

$x = 3y - 2$

$3y = x + 2$

$y = \frac{x+2}{3}$

$\boxed{f^{-1}(x) = \frac{x+2}{3} \text{ or } \frac{1}{3}x + \frac{2}{3}}$

4. $m(x) = 2\sqrt{x} - 1$

$x = 2\sqrt{y} - 1$

$x + 1 = 2\sqrt{y}$

$\sqrt{y} = \frac{x+1}{2}$

$y = \left(\frac{x+1}{2}\right)^2$

$\boxed{f^{-1}(x) = \left(\frac{x+1}{2}\right)^2}$

If you're given the inverse, you can also use this solving strategy to find the original function.

Skill #19: Polynomial Graphs

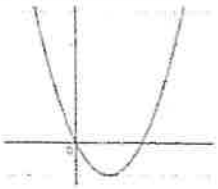
- Sketching the graph of odd and even degree polynomials. Know your end behaviors!

Note: Even and odd degree polynomials are different from even/odd FUNCTIONS. (See Skill #18.)

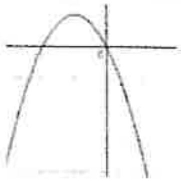
EVEN DEGREE: End behavior is either ↖ ↗ if leading coefficient (L.C.) is positive or ↙ ↘ if leading coefficient (L.C.) is negative.

Quadratic Graphs (x^2)
Degree: 2
Max # of Roots: 2

Positive L.C.:

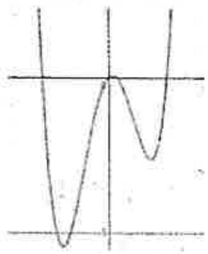


Negative L.C.

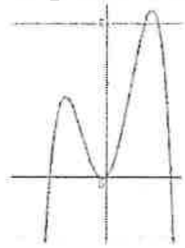


Quartic Graphs (x^4)
Degree: 4
Max # of Roots: 4

Positive L.C.:



Negative L.C.:



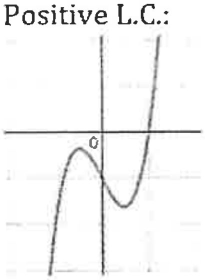
Positive L.C. End Behavior: As $x \to \infty, f(x) \to \infty$. As $x \to -\infty, f(x) \to \infty$.

Negative L.C. End Behavior: As $x \to \infty, f(x) \to -\infty$. As $x \to -\infty, f(x) \to -\infty$.

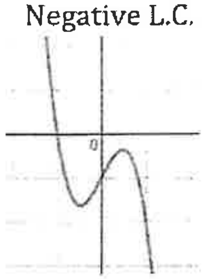
ODD DEGREE: End behavior is either ↙ ↗ if leading coefficient (L.C.) is positive or ↖ ↘ if leading coefficient (L.C.) is negative.

Cubic Graphs (x^3)
Degree: 3
Max # of Roots: 3

Positive L.C.:

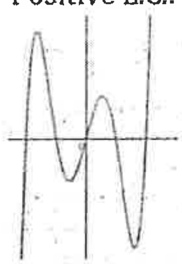


Negative L.C.

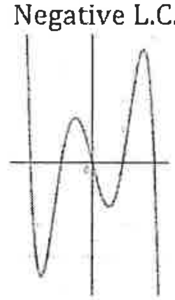


Quintic Graphs (x^5)
Degree: 5
Max # of Roots: 5

Positive L.C.:



Negative L.C.:

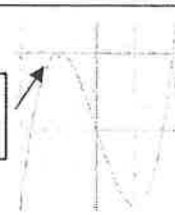


Positive L.C. End Behavior: As $x \to \infty, f(x) \to \infty$. As $x \to -\infty, f(x) \to -\infty$.

Negative L.C. End Behavior: As $x \to \infty, f(x) \to -\infty$. As $x \to -\infty, f(x) \to \infty$.

Multiplicity of A Root: How many times a root "occurs."
-If the graph crosses the x-axis at that point, the multiplicity is 1.
-If the graph bounces on the x-axis at that point, the multiplicity is 2.

Bounces at $x = -1$
Multiplicity = 2



Crosses at $x = 2$
Multiplicity = 1

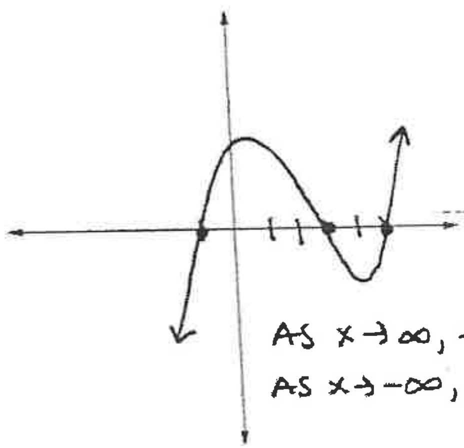
Equation: $(x+1)^2(x-2)^1$ -- The exponent represents the multiplicity.

Fill in the basic information in the chart.

Function Name	Basic Equation	Degree	At most, how many different roots?
Quadratic	$f(x) = x^2$	2	2
Cubic	$f(x) = x^3$	3	3
Quartic	$f(x) = x^4$	4	4
Quintic	$f(x) = x^5$	5	5

Sketch a basic graph given the information and describe the end behavior.

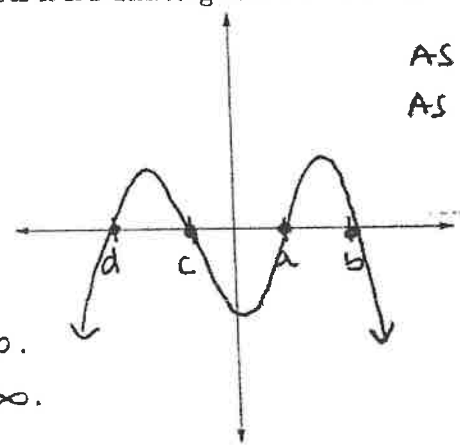
1. Cubic with roots of $-1, 3,$ and 5



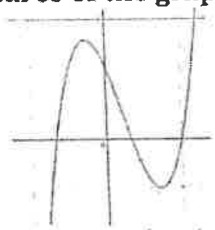
As $x \to \infty$, $f(x) \to \infty$.
As $x \to -\infty$, $f(x) \to -\infty$.

2. Negative quartic function with positive roots a & b and negative roots c & d

As $x \to \infty$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to -\infty$



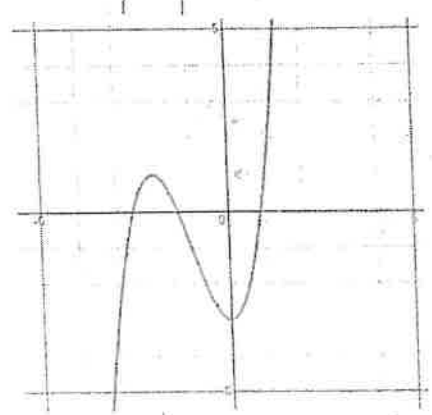
Circle the choice that best answers the questions regarding the features of the graph.



3. Which is true regarding the end behavior of the graph?
   (1) As $x \to -\infty, f(x) \to \infty$.
   (2) As $x \to 3, f(x) \to \infty$.
   (3) As $x \to \infty, f(x) \to -\infty$.
   (4) As $x \to -\infty, f(x) \to -\infty$.

4

True or False.



4. Using the graph at the right, determine if the following statements are true or false. If false, correct the statement.
   a. It is decreasing on the interval $(-2, 0)$. T
   b. It is decreasing on the interval $(0, \infty)$. F
      increasing
   c. It has a relative minimum at the point $(0, -3)$. T
   d. It has a relative maximum at the point $(0, -3)$. F
      $(-2, 1)$

Skill #20: Focus/Directrix of Parabola

- The following formula is your friend. Memorize it:

$$y = \pm \frac{1}{4p}(x-h)^2 + k$$

This formula is NOT on the reference sheet. You will have to know it.

A *parabola* is a set of points that is equidistant from a point (*focus*) and a line (*directrix*).

p represents the distance from the vertex to the focus or the directrix.

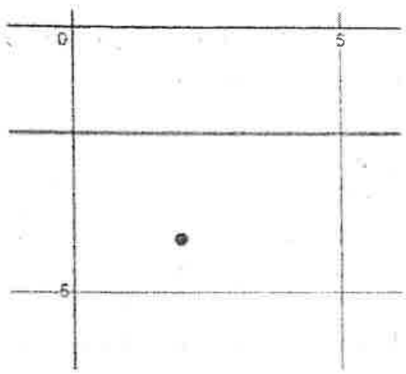
(h, k) represents the vertex. The x-value of the vertex is always the opposite sign of that in the equation.

Parabolas ALWAYS open toward the focus.

If the parabola opens upward, the coefficient is positive.

If it opens downward, the coefficient is negative.

(1) Write the equation of the parabola equidistant from:



The vertex must be between the focus and directrix, so it must be $(2, -3)$.
The distance from the vertex to the focus is 1, so $p = 1$.
Since the parabola must open toward the focus, this parabola must open downward.

Therefore, the equation must be: $y = -\frac{1}{4(1)}(x-2)^2 - 3$

$y = -\frac{1}{4}(x-2)^2 - 3$

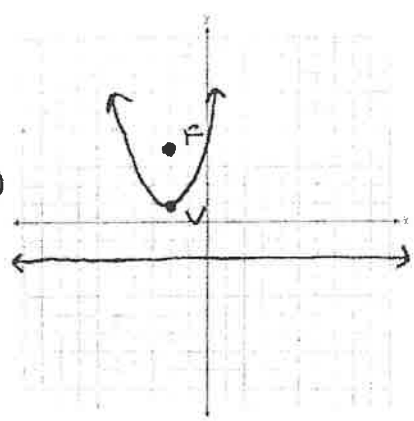
Write the equation of the parabola given the following conditions. Use the graph paper if necessary.

1. Focus: $(-2, 4)$
   Directrix: $y = -2$

2. Focus: $(3, 6)$
   Vertex: $(3, 2)$

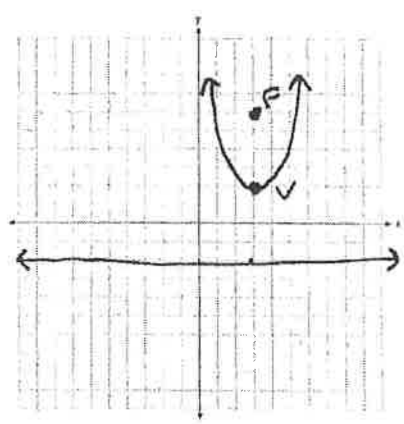
V: $(-2, 1)$

$p = 3$



$y = \frac{1}{4(3)}(x+2)^2 + 1$

$y = \frac{1}{12}(x+2)^2 + 1$



$p = 4$

$y = \frac{1}{4(4)}(x-3)^2 + 2$

$y = \frac{1}{16}(x-3)^2 + 2$

(2) Find the focus and directrix of $(x+2)^2 = 8(y-3)$.

First, solve for y.

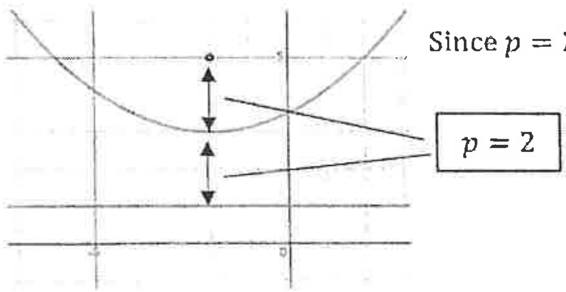
Do not expand $(x+2)^2$.

$$(x+2)^2 = 8y - 24$$
$$8y = (x+2)^2 + 24$$
$$y = \frac{(x+2)^2}{8} + \frac{24}{8}$$
$$y = \frac{1}{8}(x+2)^2 + 3$$

So the vertex is at $(-2, 3)$. Since $\frac{1}{4p} = \frac{1}{8}$, p must equal 2.

The parabola opens upward since the leading coefficient is positive.

Therefore, the parabola must look something like this:



Since $p = 2$, the focus is at $(-2, 5)$ and the directrix is at $y = 1$.

Determine the vertex, focus, and directrix of each parabola. Then sketch a quick graph.

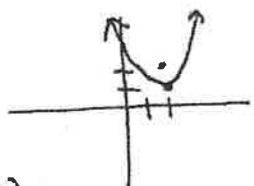
1. $y - 1 = \frac{1}{4}(x-2)^2$

$y = \frac{1}{4}(x-2)^2 + 1$

V: (2, 1)

p = 1

opens up



Focus: (2, 2)

Directrix: y = 0

2. $-8(y-2) = (x+3)^2$

$-8y + 16 = (x+3)^2$

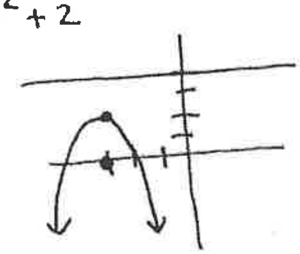
$-8y = (x+3)^2 - 16$

$y = -\frac{1}{8}(x+3)^2 + 2$

V: (−3, 2)

p = 2

opens down



Focus: (−3, 0)

Directrix: y = 4

3. $4y + x^2 = 0$

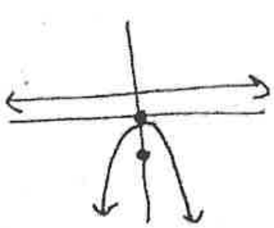
$4y = -x^2$

$y = -\frac{1}{4}x^2$

V: (0, 0)

p = 1

opens down



Focus: (0, −1)

Directrix: y = 1

4. $y - 1 = (x+5)^2$

$y = (x+5)^2 + 1$

V: (−5, 1)

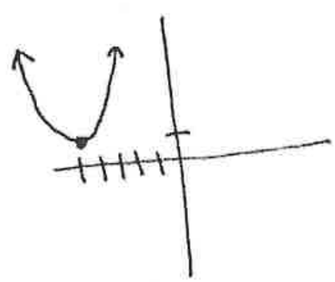
$p = \frac{1}{4}$

$\frac{1}{4p} = 1$

$4p = 1$

$p = \frac{1}{4}$

opens up



Focus: (−5, 1.25)

Directrix: y = .75

Skill #2.11: Exponential Growth/Decay

- Exponential Growth: Increasing
- Exponential Decay: Decreasing

Generic Exponential: $y = b^x$

Exponential Growth Equations:

Base	Exponent	Example
$b > 1$	*positive*	$y = 3^x$
$0 < b < 1$	*negative*	$y = \left(\frac{1}{3}\right)^{-x}$
		*This simplifies to $y = 3^x$ because the negative exponent will create the reciprocal of $\frac{1}{3}$, which is 3.

Exponential Decay Equations:

Base	Exponent	Example
$0 < b < 1$	*positive*	$y = \left(\frac{1}{2}\right)^x$
$b > 1$	*negative*	$y = 2^{-x}$
		*This simplifies to $y = \left(\frac{1}{2}\right)^x$ because the negative exponent will create the reciprocal of 2, which is $\frac{1}{2}$.

Complete the chart.

Equation	Growth or Decay?	Asymptote	End Behavior
$f(x) = 4^x$	Growth	$y = 0$	As $x \to \infty, f(x) \to \infty$ As $x \to -\infty, f(x) \to 0$
$g(x) = 2\left(\frac{1}{2}\right)^{3x}$	Decay	$y = 0$	As $x \to \infty, g(x) \to 0$ As $x \to -\infty, g(x) \to \infty$
$j(x) = e^{-x}$	Decay	$y = 0$	As $x \to \infty, j(x) \to 0$ As $x \to -\infty, j(x) \to \infty$
$k(x) = 500(0.75)^{0.77x} + 2$	Decay	$y = 2$	As $x \to \infty, k(x) \to 2$ As $x \to -\infty, k(x) \to \infty$
$h(t) = 0.88^{-t} - 4$	Growth	$y = -4$	As $t \to \infty, h(t) \to \infty$ As $t \to -\infty, h(t) \to -4$
$m(x) = \frac{1}{2}(2)^x + 15$	Growth	$y = 15$	As $x \to \infty, m(x) \to \infty$ As $x \to -\infty, m(x) \to 15$

Skill #22: Exponential Regression (Calculator Skill)

- Combo move: Stat – Edit – Enter data – Stat – Calc – 0: ExpReg

Find the exponential regression equation given the data below. Round to the *nearest hundredth.*

x	y
0	3
1	7
2	10
3	24
4	50
5	95

On you calculator, click STAT – 1:Edit and enter your data into the table.
Click STAT again, arrow over to CALC, and click 0: ExpReg.
Your Xlist should be L_1 and your Ylist should be L_2. Everything else can be left blank. Hit Calculate.

Solution: $y = a * b^x$
$a = 3.046450345$
$b = 1.988034735$

Rounding to the nearest hundredth, we have: $y = 3.05(1.99)^x$

Determine the exponential regression equations and answer the following questions.

1. Bacteria grown in a laboratory after a given number of hours is shown in the table below.

Hour	1	2	3	4	5
Bacteria	1995	2201	2430	2686	2965

Determine the exponential regression equation for this data, rounding values to the *nearest hundredth.*

$y \approx 1805.78(1.10)^x$

Assuming the exponential pattern continues, how many bacteria will there be in 8 hours? Round to the *nearest bacterium.*

$y = 1805.78(1.10)^8$
$= \boxed{3871}$

2. Determine the exponential regression equation, rounding all values to the *nearest thousandth,* given the following data table:

x	y
0	290
1	320
2	400
3	495
4	600
5	700
6	820
7	1000
8	1250
9	1580

$\boxed{y = 276.674(1.207)^x}$

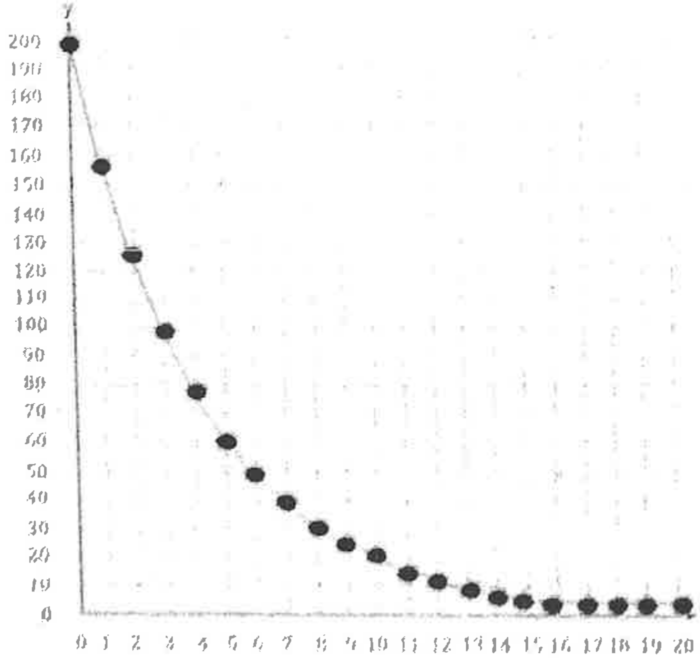
Skill #23: Exponential Graphs

- Make sure you can graph random equations, just in case they ask you to do so. It's not unheard of for your y-values to be decimals.

Given the equation: $A(t) = A_0e^{-rt}$ where $A(t)$ represents the amount of a drug left in the body after a certain amount of time, A_0 represents the initial amount of a drug in the body, r is the decay rate, and t represents time in hours.
Graph the equation when the initial dosage is 200mg and the decay rate is 0.231.

Solution: This means $A_0 = 200$ and $r = 0.231$. By substitution, we know we are graphing:
$A(t) = 200e^{-0.231t}$

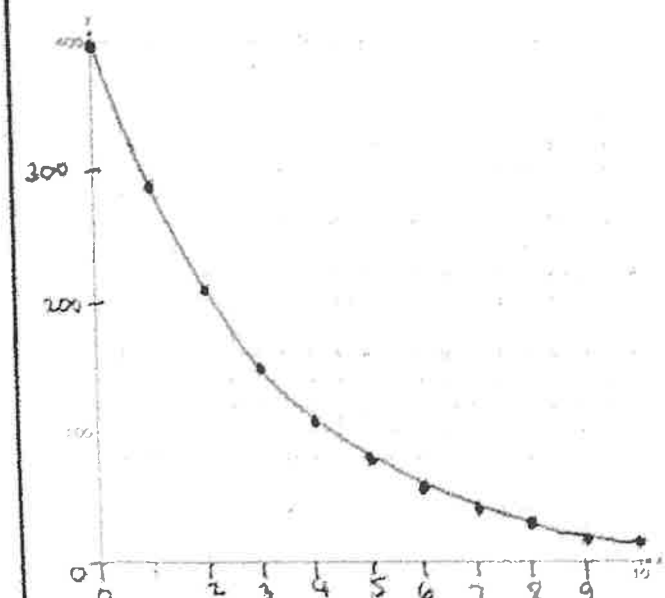
Graph this by typing it into $y_1 = 200e^{-0.231t}$
Look at the table. The highest y-value is 200. Label your axes and plot the points!

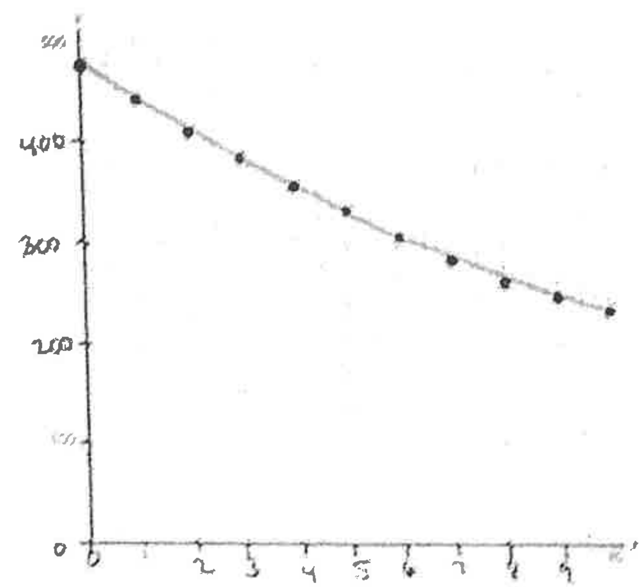


Graph the following on the axes below.

1. $y = 400(0.9)^{3x} - 2$

2. $f(x) = 475(0.75)^{.25x}$





Skill #24: Equivalent Exponential Equations

- Change an equation into an equivalent one by changing the time period.

Given $A = 24(1.075)^t$, where t is time in years, create an equation that will model the approximate:

(1) monthly growth rate (2) weekly growth rate

First, change the exponent to show that there are (1) 12 months in a year, and (2) 52 weeks in a year.

$A = 24\left(1.075^{\frac{1}{12}}\right)^{12t}$ $A = 24\left(1.075^{\frac{1}{52}}\right)^{52t}$

However, to balance our new exponent, we needed to introduce another new exponent to ensure that we are not changing the equation at all.

Since $\frac{1}{12} \cdot 12t = t$ and $\frac{1}{52} \cdot 52t = t$, we can use these as our exponents without changing the value of the equation. We will now simplify the value inside the parentheses using our calculator:

(1) $A = 24(1.006044919)^{12t}$
Since $12t$ represents m months:
$A = 24(1.006044919)^m$

(2) $A = 24(1.00139175)^{52t}$
Since $52t$ represents w weeks:
$A = 24(1.00139175)^{52}$

Complete the following conversions.

1. An antique appreciates according to the equation $f(t) = 500(1.1)^t$, where t is time in years. Determine an equivalent equation that would model the approximate monthly growth rate.

$500(1.1^{\frac{1}{12}})^{12t}$

$500(1.00797414)^{12t}$ $\boxed{f(m) = 500(1.00797414)^m}$

2. The population of a city depreciates according to the equation $y = 25000(0.96)^x$, where x is time in years. Determine an equivalent equation that would model the approximate weekly growth rate.

$25000(.96^{\frac{1}{52}})^{52t}$

$25000(.9992152697)^{52t}$ $\boxed{y = 25000(.9992152697)^w}$

Multiple Choice. Circle the choice that best answers the question.

3. The amount of visitors to a national park has grown according to the model $P = 3000(1.21)^t$, where t is the time in years. Which of the following equations can model the approximate monthly growth rate in terms of m?

2

(1) $P = 3000(0.101)^m$
(2) $P = 3000(1.016)^m$
(3) $P = 3000(0.101)^{12m}$
(4) $P = 3000(1.016)^{12m}$

$(1.21^{\frac{1}{12}})^{12t}$

$(1.016011868)^m$

Skill #25: Use and Apply Mortgage Formulas

- Big formulas are no sweat! (Note: It might not always be the formula below.)
- Down Payment: Amount of money paid initially, usually used toward a house or car. The rest of the money needed to buy a house or car usually comes from a loan.

Loren wants to buy a new home for $162,700 near his favorite city. The formula to compute a mortgage payment, M, is $M = P \cdot \frac{r(1+r)^N}{(1+r)^N - 1}$ where P is the principal amount of the loan, r is the monthly interest rate, and N is the number of monthly payments. Loren's bank offers a monthly interest rate of 0.205% for a 15-year mortgage.

(1) With no down payment, determine Loren's mortgage payment to the *nearest dollar.*

$N = 15 \cdot 12 = 180$

$$M = 162{,}700 \cdot \frac{0.00205(1 + 0.00205)^{180}}{(1 + 0.00205)^{180} - 1} = \$1{,}082$$

(2) Algebraically determine and state the down payment, rounded to the *nearest cent,* that Loren needs to make in order for his mortgage payment to be $1000.

$$1000 = P \cdot \frac{0.00205(1 + 0.00205)^{180}}{(1 + 0.00205)^{180} - 1}$$

$1000 = 0.0066490789P$

$P = \$150{,}396.77 \leftarrow$ This is your loan amount.

So, your down payment was: $162{,}700 - 150{,}396.77 = \$12{,}303.23$

Using the equation from above, answer the following questions.

1. Find the mortgage payment for a house that costs $280,000, assuming a down payment of $50,000, a mortgage rate of 0.333% for a 30-year mortgage. Round to the *nearest cent.*

$$\begin{array}{r} 280000 \\ -\ 50000 \\ \hline 230000 \end{array} \rightarrow P$$

$r = .00333$

$N = 30(12) = 360$

$$M = 230000 \cdot \frac{.00333(1.00333)^{360}}{(1.00333)^{360} - 1}$$

$$= \boxed{\$1097.52}$$

2. Determine the down payment needed in order for a mortgage payment to be $1200. Assume the house costs $152,000 at 0.625% monthly interest for 20 years. Round to the *nearest dollar.*

$r = .00625$

$N = 20(12) = 240$

$P = ?$

$$1200 = P \cdot \frac{.00625(1.00625)^{240}}{(1.00625)^{240} - 1}$$

$1200 = P(.0080559319)$

$P = 148958.56$

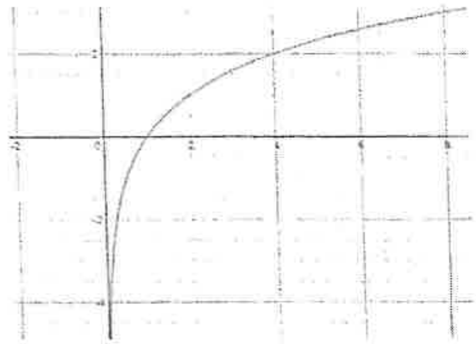
down payment $= 152000 - 148958.56 = \boxed{\$3041}$

Skill #26: Properties of Logarithms and Their Graphs

- Logarithms and exponentials are inverses!

What is the inverse of $y = 3^x$?
$x = 3^y$ Switch x and y.
$\log_3 x = y$ Solve for y.
$f^{-1}(x) = \log_3 x$

What is the inverse of $y = \log_2 x$?
$x = \log_2 y$ Switch x and y.
$2^x = y$ Solve for y.
$f^{-1}(x) = 2^x$



Generic Graph of Logarithm:
There is an asymptote at $x = 0$.
Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

End Behavior:
As $x \to \infty, f(x) \to \infty$.
As $x \to 0, f(x) \to -\infty$.
(Note: x cannot approach $-\infty$ because of the asymptote.)

Find the inverse of each of the following.

Function	Inverse
$f(x) = \log_3 x$	$y = 3^x$
$f(x) = \log_{\frac{1}{3}} x$	$y = (\frac{1}{3})^x$
$f(x) = 6^x$	$y = \log_6 x$
$f(x) = e^x$	$y = \ln x$

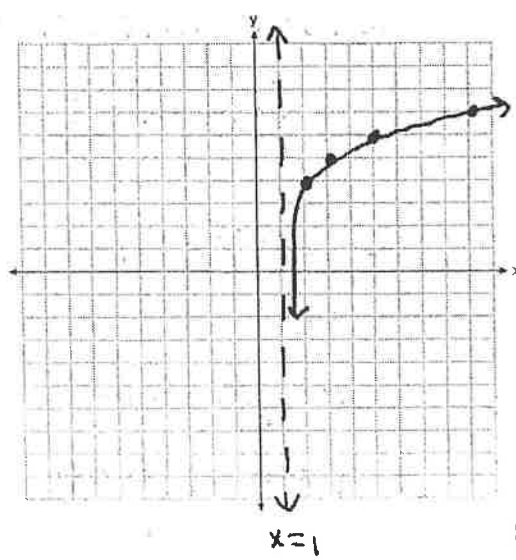
Answer the following questions regarding graphs of logarithms. Use the graph paper for #2.

1. The graph of $y = \log_2 x$ is translated to the left 1 unit and down 3 units. What is the equation of the translated graph?

$y = \log_2(x+1) - 3$

2. Graph $y = \log_2(x-1) + 4$ on the graph paper. Describe the end behavior.

AS $x \to \infty$, $f(x) \to \infty$.
AS $x \to 1$, $f(x) \to -\infty$



Skill #27: Solving Exponential Equations Using Logarithms

- You must know this conversion: $3^x = 8 \leftrightarrow \log_3 8 = x$

Common Logarithm: $\log_{10} x$ --- often written without the base --- $\log x$
Natural Logarithm: $\log_e x$ --- often written with different notation --- $\ln x$
No matter what type of base you have, they all are approached in the same way!

Solve for x to the *nearest tenth.*

$8(2^{x+7}) + 3 = 37$	First, isolate the exponential. In this case, it is 2^{x+7}.
$8(2^{x+7}) = 34$	Subtract 3 on both sides.
$2^{x+7} = 4.25$	Divide by 8. (Now the exponential is isolated.)
$\log_2 4.25 = x + 7$	Convert into logarithmic form.
$x = \log_2 4.25 - 7$	Subtract 7.
$x = -4.9$	Evaluate using your calculator. (Different log bases can be found by clicking MATH – A:logBASE.)

Note: The answer of $\log_2 4.25 - 7$ is called the *exact* answer since it is not rounded at all.

Solve each of the following equations.

1. Solve to the *nearest tenth:* $4 \cdot 3^n + 15 = 359$

$4 \cdot 3^n = 344$

$3^n = 86$

$\log_3 86 = n$

$\boxed{n = 4.1}$

2. Solve $\frac{87e^{0.3x}}{87} = \frac{5918}{87}$ to the *nearest thousandth.*

$e^{.3x} = 68.02298851$

$\ln 68.02298851 = .3x$

$\boxed{x = 14.066}$

3. Solve $6 \cdot 16^{7y+2} - 3 = 81$ to the *nearest hundredth.*

$6 \cdot 16^{7y+2} = 84$

$16^{7y+2} = 14$

$\log_{16} 14 = 7y + 2$

$y = \frac{\log_{16} 14 - 2}{7}$

$\boxed{y = -.15}$

Skill #28: Logarithmic Word Problems

- Many times you are asked to apply formulas that are not given. The following are the ones to memorize.

  Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

  Continuous Compound Interest: $A = Pe^{rt}$

  Half-Life: $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

$1200 was placed in a bank account and, after a certain number of years, there was $3000 in the account.

a. How many years have passed if the interest was compounded quarterly at 4.25%?

$$3000 = 1200\left(1 + \frac{0.0425}{4}\right)^{4t}$$

$2.5 = \left(1 + \frac{0.0425}{4}\right)^{4t}$	Divide by 1200.
$2.5 = (1.010625)^{4t}$	Simplify inside parentheses.
$\log_{1.010625} 2.5 = 4t$	Convert to log form.
$t = \frac{\log_{1.010625} 2.5}{4}$	Divide by 4.
$t \approx 21.7$ years	Evaluate.

b. How many years have passed if the interest was compounded continuously at 3%?

$$3000 = 1200e^{0.03t}$$

$2.5 = e^{0.03t}$	Divide by 1200.
$\ln 2.5 = 0.03t$	Convert to log form.
$t = \frac{\ln 2.5}{0.03}$	Divide by 0.03.
$t \approx 30.5$ years	Evaluate.

The half-life of a certain compound is 4 days. If the initial amount of the compound was 100g and now there is 62.4g, how many days have passed?

$$62.4 = 100\left(\frac{1}{2}\right)^{\frac{t}{4}}$$

$0.624 = \left(\frac{1}{2}\right)^{\frac{t}{4}}$	Divide by 100.
$\log_{1/2} 0.624 = \frac{t}{4}$	Convert to log form.
$4\log_{1/2} 0.624 = t$	Multiply by 4.
$t \approx 2.7$ days	Evaluate.

Compound Interest.

1. Nevaeh's parents gave her $2,500 to invest for her 18th birthday. She is considering an investment option that will pay her 3.5% compounded monthly. Algebraically determine, to the *nearest tenth of a year,* how long it would take for this option to double Nevaeh's investment.

$$5000 = 2500\left(1 + \frac{.035}{12}\right)^{12t}$$

$$2 = (1.002916667)^{12t}$$

$$\log_{1.002916667} 2 = 12t$$

$$\boxed{t = 19.8\text{ yr}}$$

2. Joanna deposited $1,000 at 2.8% interest compounded weekly. In how many years, to the *nearest tenth*, will she have $11,000 in the account?

$$11000 = 1000\left(1 + \frac{.028}{52}\right)^{52t}$$

$$11 = (1.000538462)^{52t}$$

$$\log_{1.000538462} 11 = 52t$$

$$\boxed{t = 85.7}$$

Continuous Compound Interest.

3. After how many years will $100, invested at an annual interest rate of 4% compounded continuously, be worth $450? Round to the *nearest tenth.*

$$450 = 100e^{.04t}$$

$$4.5 = e^{.04t}$$

$$\ln 4.5 = .04t$$

$$t = \frac{\ln 4.5}{.04} = \boxed{37.6}$$

4. In 2000, there was an influx of a new species of insect in a local park. This new insect population is growing continuously at a rate of 8% per year. If the park initially had 50 new insects, in what year will there be three times that number?

$$150 = 50e^{.08t}$$

$$3 = e^{.08t}$$

$$\ln 3 = .08t$$

$$t = \frac{\ln 3}{.08} = \boxed{13.73265361}$$

Half-Life.

5. Sodium iodide-131, used to treat certain medical conditions, has a half-life of 1.8 hours. A patient took a 150 mcg dose of sodium iodide-131. Determine, to the *nearest tenth of an hour*, how long it will take before the amount in her body will reduce to 30 mcg.

$$30 = 150\left(\tfrac{1}{2}\right)^{\frac{t}{1.8}}$$

$$.2 = \left(\tfrac{1}{2}\right)^{t/1.8}$$

$$\log_{\frac{1}{2}} .2 = \frac{t}{1.8}$$

$$t = 1.8\log_{\frac{1}{2}}(.2)$$

$$\boxed{t = 4.2}$$

6. A given substance has a half-life of 6,000 years. After t years, one-fifth of the original sample remains radioactive. Find t to the *nearest thousand years.*

$$\frac{1}{5} = \left(\tfrac{1}{2}\right)^{t/6000}$$

$$\log_{\frac{1}{2}}\left(\tfrac{1}{5}\right) = \frac{t}{6000}$$

$$t = 6000\log_{\frac{1}{2}}\left(\tfrac{1}{5}\right)$$

$$t = 13931.56857$$

$$\boxed{t = 14000}$$

Skill #29: Recursive Formulas for Sequences

- These are formulas that are based on the previous term. They are usually written in terms of a_{n-1}. You must include the first term, a_1!

Arithmetic:
Write a recursive formula for: $4, -1, -6, -11, \ldots$
Solution: $a_1 = 4$ You must include the first term!
$a_n = a_{n-1} - 5$

Take the **previous term** and subtract 5.

Geometric:
Write a recursive formula for: $3, 4.5, 6.75, 10.125, \ldots$
Solution: $a_1 = 3$ You must include the first term!
$a_n = 1.5a_{n-1}$

Multiply the **previous term** by 1.5.

Neither Arithmetic Nor Geometric:
Write a recursive formula for:
$2, 5, 11, 23, \ldots$
Solution: $a_1 = 2$
You must include the first term!
$a_n = 2a_{n-1} + 1$

Take the **previous term** and multiply it by 2, then add 1.

Identify if the sequence is arithmetic, geometric, or neither. Write a recursive formula for each of the following sequences.

1. $-4, -6, -8, -10, \ldots$ arithmetic
$a_1 = -4$
$a_n = a_{n-1} - 2$

2. $19, 13, 7, 1, \ldots$ arithmetic
$a_1 = 19$
$a_n = a_{n-1} - 6$

3. $25, 75, 225, \ldots$ geometric
$a_1 = 25$
$a_n = 3a_{n-1}$

4. $3, 9, 27, \ldots$ geometric
$a_1 = 3$
$a_n = 3a_{n-1}$

5. $-1, -4, -13, \ldots$ neither
$a_1 = -1$
$a_n =$

6. $1, 5, 17, 53, \ldots$ neither
$a_1 = 1$
$a_n = 3a_{n-1} + 2$

Application.

7. At her job, Monica earns \$35,000 the first year and receives a raise of \$1,500 each successive year. Write a recursive formula that will model her salary.
$a_1 = 35000$ $a_n = a_{n-1} + 1500$

8. Find the first 5 terms of the sequence: $a_1 = 2, a_n = 2a_{n-1} + 5$
2 9 23 51 107
$2(2)+5$ $2(9)+5$ $2(23)+5$ $2(51)+5$

Skill #30: Explicit Formulas for Sequences

- These formulas are given to you on the Regents.

Arithmetic: $a_n = a_1 + (n-1)d$ Geometric: $a_n = a_1 r^{n-1}$

Arithmetic sequences are those in which each successive term is created by adding or subtracting a certain amount. This amount is called the *common difference, d.*

Write an explicit formula for the sequence: $4, -1, -6, -11, \ldots$

Solution: Since each term is created by subtracting 5, then $d = -5$.
Substituting into the formula, we have:

$a_n = 4 + (n-1)(-5)$

$a_n = 4 - 5n + 5$ Distribute the -5.

$a_n = 9 - 5n$ Simplify.

Geometric sequences are those in which each successive term is created by multiplying by a certain amount. This amount is called the *common ratio, r.*

Write an explicit formula for the sequence: $3, 4.5, 6.75, 10.125, \ldots$

Solution: Since each term is created by multiplying by 1.5, then $r = 1.5$.
Substituting into the formula, we have:

$a_n = 3(1.5)^{n-1}$

There are some ways to simplify this, but, for the most part, you don't see it simplified often.

Identify if the sequence is arithmetic or geometric. Write an explicit formula for each of the following sequences. If the sequence is arithmetic, be sure to simplify your formula completely.

1. $-2, 4, 10, 16, \ldots$

arithmetic

$a_n = -2 + (n-1)(6)$

$a_n = -2 + 6n - 6$

$a_n = 6n - 8$

2. $4, 10, 25, 62.5, \ldots$

geometric

$a_n = 4(2.5)^{n-1}$

3. $14, 3, -8, -19, \ldots$

arithmetic

$a_n = 14 + (n-1)(-11)$

$a_n = 14 - 11n + 11$

$a_n = -11n + 25$

4. $63, 21, 7, \frac{7}{3}, \ldots$

geometric

$a_n = 63(\frac{1}{3})^{n-1}$

Application.

5. Monica deposited 1 cent into a bank account on the first day of the month. She then deposited 3 cents on the second day, 9 cents on the third day, and 27 cents on the fourth. Assuming the pattern continues, write an explicit formula to represent this scenario.

$a_n = .01(3)^{n-1}$

6. Given an arithmetic sequence with $a_3 = -9$ and $a_7 = 7$, determine the common difference.

$\underset{a_3}{\underline{-9}}$ ___ ___ ___ $\underset{a_7}{\underline{7}}$

$-9 + 4d = 7$

$4d = 16$

$\boxed{d = 4}$

7. Given a geometric sequence with $a_2 = 6$ and $a_6 = 1536$, determine two values for the common ratio.

$\underset{a_2}{\underline{6}}$ ___ ___ ___ $\underset{a_6}{\underline{1536}}$

$6r^4 = 1536$

$r^4 = 256$

$\boxed{r = \pm 4}$

Skill #31: Summation Formulas for Sequences (Series)

- Quickest way to take a sum is to use the summation (Greek sigma) Σ button on your calculator. For geometric series, there is also this formula given to you on the reference sheet: $S_n = \frac{a_1 - a_1 r^n}{1-r}$

Find the sum of the first 10 terms of the sequence: $4, -1, -6, -11, \ldots$

Since this sequence is arithmetic, the sigma notation will be the only method we will use to take the sum.

First, find the term formula:

$$a_n = a_1 + (n-1)d$$
$$a_n = 4 + (n-1)(-5)$$
$$a_n = 9 - 5n$$

Then, use the summation symbol around the term formula:

last term

$$S_{10} = \sum_{n=1}^{10} (9 - 5n) = -185$$

first term

Note: The sigma symbol can be found under MATH --- 0:summation

Find the sum of the first 12 terms of the sequence: $2, -6, 18, -54, \ldots$

Since this sequence is geometric, we can use two different methods to solve.

Method 1: sigma notation

First, find the term formula:

$$a_n = a_1 r^{n-1}$$
$$a_n = 2(-3)^{n-1}$$

Then, use the summation symbol around the term formula:

last term

$$S_{12} = \sum_{n=1}^{12} (2(-3)^{n-1}) = -265720$$

first term

Method 2: formula from reference sheet

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_{12} = \frac{2 - 2(-3)^{12}}{1 - (-3)} = -265720$$

Determine whether your sequence is arithmetic or geometric. Then find the sum using an appropriate method.

1. Find the summation of $6, 1, -4, -9, \ldots$ through 11 terms.

$a_n = 6 + (n-1)(-5)$
$= 6 - 5n + 5$
$= 11 - 5n$

$S_{11} = \sum_{n=1}^{11} (11 - 5n) = \boxed{-209}$